(GRAPH) QUANTUM MECHANICS: TOWARDS A HOMOTOPICAL APPROACH TO SPECTRAL GRAPH THEORY

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1. Introduction

The goal of these notes is to show how standard homotopical techniques reproduce many physically interesting numerical invariants of scalar field theory on a graph. That is, we will show how to compute the higher cumulants of this model. In some sense, these notes are an example of how to do probability theory algebraically.

We warn the reader that throughout our discussion, our mathematical constructions (such kernels etc.) should be interpreted in a homotopical sense. For the reader's convenience, we will express these at the point-set level as cochain complexes. In particular, connective objects are contained in negative degrees, and the differentials are degree 1.

One can reasonably argue that this approach is overkill: the mathematics is much more "advanced" than it needs to be. For example, we will assume a basic familiarity with homotopy theory. The author hopes that the basic ingredients are simple enough that these notes may serve as an entrance for contemporary applications of homotopy theory to field theory via the Batalin-Vilkovisky forrmalism.

2. Input and Notation

Our basic input will be an undirected weighted graph, which we'll denote as Γ. Throughout, we will use the following notation:

- Γ_0 will denote the set of vertices.
- Γ_1 will denote the set of edges.
- $\mathrm{w}_\Gamma : \Gamma_1 \to (0, \infty)$ will denote the weight function.
- $|\Gamma|$ the obvious space associated to the graph Γ, i.e. it's geometric realization.

Moreover, given a directed edge $e : \Delta^1 \to \Gamma$, we will denote it's ith vertex as e_i.

3. The Graph Laplacian

In this section, we perform a standard maneuvre from spectral graph theory: encode aspects of this weighted graph as a linear object.

We will be to linearize the above combinatorial data by constructing a linear operator we will refer to as the graph Laplacian. This will be built from the following (manifestly symmetric) pairing:

$$\Gamma_0^{\times 2} \longrightarrow \mathbb{R} \tag{3.1}$$

$$\left(v : * \coprod * \to \Gamma\right) \longmapsto -\frac{1}{2} \sum_{\substack{\gamma : e \to \tilde{\Gamma} \\ e|_{* \coprod *} = v}} (-1)^{\delta_{e_0, e_1}} \mathrm{w}_\Gamma(e) \tag{3.2}$$

Remark 1. Away from the diagonal it is a weighted sum over edges connecting the two chosen vertices.

Definition 2. Taking the adjoint of the linearization of the above pairings defines the *graph Laplacian* associated to Γ:

$$\mathrm{Q}_\Gamma : \mathbb{R}^{\Gamma_0} \to (\mathbb{R}^{\Gamma_0})^\vee$$

Remark 3. We are thinking $\mathbb{R}^{\Gamma_0}$ as the vector space of functions from the set of vertices to $\mathbb{R}$. In other words, a point in this space is a decoration of the vertices by real numbers.

Remark 4. The set of vertices give a natural basis for $\mathbb{R}^{\Gamma_0}$. We invite the reader to imagine one of these vector as representing some "body" concentrated at that vertex.

In the canonical basis spanned by the vertices $\{\phi_i\}$, we can suggestively express Q_Γ as a vector field with linear coefficients:

$$\mathrm{Q}_\Gamma = \frac{1}{2} Q_{ij} \phi_j^\dagger \frac{\partial}{\partial \phi_i}$$

where we adopt the notation that $\phi_i^\dagger$ is the linear coordinate associated to ϕ_i.

Remark 5. Note that $Q_{ij} = Q_{ji}$, so that (Q_{ij}) defines a symmetric matrix.

We may model aspects of the above as a single linear object, by taking the kernel of this map. This admits a standard representation as a single cochain complex:

$$\mathcal{E}_\Gamma := \left(\mathbb{R}^{\Gamma_0} \xrightarrow{\mathrm{Q}_\Gamma} (\mathbb{R}^{\Gamma_0})^\vee[-1]\right) = \ker(\mathrm{Q}_\Gamma)$$

As $\mathcal{E}_\Gamma$ is the kernel of a map out of $\mathbb{R}^{\Gamma_0}$, there is a natural linear map:

$$\mathcal{E}_\Gamma \to \mathbb{R}^{\Gamma_0}$$

Remark 6. In order to think of this geometrically, we invite the reader to view this object geometrically. This is accomplished by viewing $\mathcal{E}_\Gamma$ through Grothendieck's S-point formalism. For computational reasons, we will study this object perturbatively, around the zero vector of $\mathcal{E}_\Gamma$. The theory of formal moduli problems provides a rigorous and principled setting to perform these computations. That is, we'll view $\mathcal{E}_\Gamma$ via its

$$\begin{aligned} \mathrm{CAlg}^{\mathrm{aug}} &\longrightarrow \mathcal{S}\mathrm{paces} \\ A &\longmapsto \mathrm{Maps}_{\mathrm{Vect}}(\mathcal{E}_\Gamma^\vee, A) \end{aligned}$$

Furthermore, the diagonal terms of Δ_Γ were designed so that there is an equivalence

$$\mathcal{E}_\Gamma \simeq \mathbb{R}^{\pi_0(|\Gamma|)} \oplus \left(\mathbb{R}^{\pi_0(|\Gamma|)}\right)^\vee[-1]$$

Therefore, $\mathcal{E}_\Gamma$ encodes the (-1)-shifted cotangent bundle of the connected components of $|\Gamma|$. Therefore, this linear object only encodes the connected components of the graph.

We will describe how to encode certain geometric features through a certain nondegenerate pairing. We will present this pairing in two, dual forms: a symmetric form and a symplectic forms. Their duality is explicitly related through Koszul duality.

3.1. **Symplectic Formulation.** The above object has additional structure: a symplectic form of degree -1, induced by the pairing between $\mathbb{R}^{\Gamma_0}$ and $(\mathbb{R}^{\Gamma_0})^\vee$:

$$\omega_\Gamma : \Lambda^2(\mathcal{E}_\Gamma) \to \mathbb{R}[-1]$$

so that $\omega_\Gamma \in \Lambda^2(\mathcal{E}_\Gamma^\vee)[-1]$. Linear coordinates provide Darboux coordinates for this form:

$$\omega_\Gamma = \sum_{i \in \Gamma_0} \mathrm{d_{dR}}\phi_i \wedge \mathrm{d_{dR}}\phi_i^\dagger$$

The nondegeneracy means that contracting ω_Γ along a vector field induces an equivalence:

$$\omega_\Gamma^\vee : \mathcal{E}_\Gamma \simeq \mathcal{E}_\Gamma^\vee[-1]$$
$$X \mapsto \iota_X \omega$$

Throughout our discussion, we adopt standard notation in the BV formalism. A generic element of $\mathbb{R}^{\Gamma_0}$ will be denoted ϕ. Given any ϕ, it's conjugate momentum will be denoted $\varphi^\dagger$. In other words, our notation is such that:

$$\omega_\Gamma^\vee(\phi) = \phi^\dagger$$

In summary, we have encoded aspects of the graph Γ into an $\mathbb{R}-$linear object $\mathcal{E}_\Gamma$, equipped with a symplectic form of degree -1, ω_Γ.

Note that $\mathcal{E}_\Gamma$

3.2. **Symmetric Formulation.** As an aside, we point out that abstract nonsense shows $\omega_\Gamma^\vee$ translates the data of ω_Γ to a map of the form:

$$< -, - >: \mathrm{Sym}^2\left((\mathrm{T}[-1]|_{\phi=0}\mathcal{E}_\Gamma)^\vee\right) \to \mathbb{R}[3]$$

In other words, the symplectic form on $\mathcal{E}_\Gamma$ of degree (-1) is equivalent to a nondegenerate symmetric pairing of degree $2-(-1)=3$ on $\mathrm{T}[-1]|_0\mathcal{E}_\Gamma \simeq \mathcal{E}_\Gamma[-1]$.

We now describe the Poisson bracket associated to ω_Γ.

3.3. **Poisson Bracket.** First, the equivalence $\omega_\Gamma^\vee$ provides an antisymmetric linear pairing on $\mathcal{E}_\Gamma^\vee[-1]$:

$$\Lambda^2(\mathcal{E}_\Gamma^\vee[-1]) \to \mathbb{R}[-1]$$

whose kernel $\{-,-\} \in \Lambda^2(\mathcal{E}_\Gamma[1])$

The general, and somewhat surprising isomorphism:

$$\Lambda^2(\mathcal{E}_\Gamma^\vee[-1]) \simeq \mathrm{Sym}^2(\mathcal{E}_\Gamma^\vee)[-2]$$

means that, for a (-1) shifted symplectic structure, the Poisson kernel is a *symmetric*

$$\mathrm{K}_\Gamma = \frac{\partial}{\partial a_i}\frac{\partial}{\partial a_i^\dagger} \in \mathrm{Sym}^2(\mathcal{E}_\Gamma)[1]$$

Viewing this as a formal second order differential operator yields an endomorphism of polynomials in $\mathcal{E}_\Gamma$ of degree 1 and weight -2:

$$\partial_{\mathrm{K}_\Gamma} : \mathrm{Sym}^\bullet(\mathcal{E}_\Gamma^\vee) \to \mathrm{Sym}^{\bullet-2}(\mathcal{E}_\Gamma^\vee)$$

which squares to zero, reflecting that partial derivatives commute.

In order to illuminate the relationship of the above object with field theory, we now show how to encode the above data in a generating function, which we'll refer to as

3.4. **The Classical Action.** Note that we can combine the pairing and Q_Γ into a quadratic function:

$$\mathrm{S}_\Gamma(\phi, \phi^\dagger) = \frac{1}{2}\omega_\Gamma(\phi, \mathrm{Q}_\Gamma \phi)$$

So that S_Γ has degree zero. [1] In the the standard basis, one may compute:

$$\mathrm{S}_\Gamma(\phi, \phi^\dagger) = \frac{1}{2}\sum_{e\in\Gamma_1} \mathrm{w}_\Gamma(e)(\phi(e_0) - \phi(e_1))^2$$

So that S_Γ gives a (weighted) measure of how ϕ changes between adjacent vertices. We shall refer to this action as the *classical action.*

Explicit calculation shows that the action is the generating function for the net weight between distinct vertices. More precisely,

$$\frac{\partial^2}{\partial a_i \partial a_j}|_{\phi=0} S$$

is the sum of weights of all the (undirected) edges connecting i and j. In particular, the classical action is "local" in the sense that:

$$\frac{\partial^2}{\partial a_i \partial a_j}\mathrm{S}_\Gamma = 0$$

whenever there does not exist an edge connecting i to j.

Remark 7. The action also has a clear interpretation in terms of the symplectic structure. First, one can see that:

$$\mathrm{d}_{\mathrm{dR}}\mathrm{S}_\Gamma = \iota_{\mathrm{Q}_\Gamma}\omega_\Gamma$$

Borrowing the language of symplectic geometry, S_Γ is a Hamiltonian function for the vector field associated to Q_Γ. Equivalently, S_Γ is the conserved quantity associated to the symmetry generated by the vector field Q_Γ:

$$\{S, -\} = \mathrm{Q}_\Gamma$$

[1] the degree of Q_Γ compensates for the shift of ω_Γ

Moreover, one can see that $\mathcal{E}_\Gamma$ is the pullback of:

$$\mathbb{R}^{\Gamma_0} \xrightarrow{d(0)} \mathrm{T}^*\mathbb{R}^{\Gamma_0} \xleftarrow{\mathrm{d_{dR}S_\Gamma}} \mathbb{R}^{\Gamma_0}$$

In other words, $\mathcal{E}_\Gamma$ is the intersection of $\mathrm{d_{dR}S_\Gamma}$ with the zero section, a.k.a. the critical locus of S. In other words, the action provides a variational formulation for $\mathcal{E}_\Gamma$

3.5. **Diagonalization.** We now exploit the spectral theorem in order to simultaneously split the $\mathcal{E}_\Gamma$ and it's symmetric pairing K_Γ. More precisely, the classical spectral theorem first states that there exists an equivalence:

$$A : \mathcal{E}_\Gamma \simeq \bigoplus_{\lambda_i \in \mathrm{Spec}(\Delta_\Gamma)} \Big(\mathbb{R}\{p_i\} \cdot \xrightarrow{Q_{p_i}} \mathbb{R}\{p_i^\dagger\}[-1] \Big)$$

Where:

$$Q_p = \lambda_i p_i^\dagger \frac{\partial}{\partial p_i}$$

So that (λ_i, p_i) form an "eigenbasis" with $\lambda_i \geq 0$.

The second part of the spectral theorem states that this eigenbasis may be chosen to be "orthonormal" with respect to our pairing, In this context, this means we can chose A so that

$$A^*\mathrm{K}_\Gamma = \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_i^\dagger}$$

Reflecting that symmetric matrices admit an orthogonal eigendecomposition.

Remark 8. This diagonalization is arguably the primary benefits of this approach. For example, every positive number gives a projection map which collapses eigenspaces with eigenvalues greater than the chosen number. Geometrically, this transformation efficiently filters out decorations which vary along edges beyond a threshold set by our chosen number. This compression is very hard to articulate in the language of graphs: the linearization was essential.

Therefore, in order to exploit the above diagonalization, the invariance of our constructions is essential.

4. Summary So Far

So far, we have constructed from the data of a weighted graph Γ the following:

- A cochain complex $\mathcal{E}_\Gamma$, equipped with a map:

$$\mathcal{E}_\Gamma = \ker(\mathrm{Q}_\Gamma) \to \mathbb{R}^{\Gamma^0}$$

- A (-1) shifted symplectic form, $\omega_\Gamma : \Lambda^2(\mathcal{E}_\Gamma^\vee)[-1]$, giving rise to an equivalence:

$$\omega_\Gamma^\vee : \mathcal{E}_\Gamma \simeq \mathcal{E}_\Gamma^\vee[-1]$$

  Which we can use to write down a pairing encoding the Poisson bracket
- $\{-,-\} : \Lambda^2(\mathcal{E}_\Gamma^\vee[-1]) \to \mathbb{R}\cdot\hbar[-1]$ with associated Poisson Kernel $\mathrm{K}_\Gamma \in \mathrm{Sym}^2(\mathcal{E}_\Gamma)[-1]$. This may be equivalently seen as:

$$\Delta_\Gamma : \mathrm{Sym}^2\big((\mathcal{E}_\Gamma^\vee[-1])[1]\big) \to \mathbb{R}[1]$$

With these classical ingredients in place, we'd now like to execute a standard

5. Quantization Procedure

Dirac's quantization recipe tells us that we should interpret the Poisson bracket on linear functions on a symplectic vector space as a Lie extension.

This follows from first projecting off the higher than quadratic parts of the underlying

$$\mathcal{U}_{\mathrm{Mod_k}}\mathrm{CE}_*(\mathcal{E}_\Gamma^\vee[-1]) \to \mathrm{Sym}^2(\mathcal{E}_\Gamma^\vee[-1]) \overset{\Delta_\Gamma}{\to} \mathbb{R}[1]$$

Which, through the procedure outlined in the appendix, produces a central extension of $\mathcal{E}_\Gamma^\vee[-1]$ of degree 1 - 2 = -1.

We'll refer to this central extension as the Heisenberg Lie algebra associated to $(\mathcal{E}_\Gamma, \omega_\Gamma)$, $\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)$. By definition, it fits into a fibre sequence of Lie algebras:

$$\mathbb{R}\cdot\hbar[-1] \longrightarrow \mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma) \overset{\hbar=0}{\longrightarrow} \mathcal{E}_\Gamma^\vee[-1]$$

Therefore, we think of the Poisson bracket as endowing $\mathcal{E}_\Gamma^\vee[-1]\oplus\mathbb{R}\cdot\hbar[-1]$ with the structure of a Lie algebra, which we will denote:

$$\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma) \simeq \Big((\mathcal{E}_\Gamma^\vee \oplus \mathbb{R}\cdot\hbar)[-1], [-,-] = \hbar\{-,-\}\Big)$$

We can take the Chevalley-Eilenberg chains functor to this sequence, to obtain a sequence of coalgebras:

$$\mathbb{R}[\hbar] \to \mathrm{CE}_*(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)) \overset{\hbar=0}{\longrightarrow} \mathrm{Sym}^\bullet(\mathcal{E}_\Gamma^\vee)$$

Our previous discussion provides an explicit model for the middle term in the above sequence as the cochain complex:

$$\mathrm{CE}_*(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)) \simeq \Big(\mathrm{Sym}(\mathcal{E}_\Gamma^\vee)[\hbar], \mathrm{Q}_\Gamma + \hbar\partial_{\mathrm{K}_\Gamma}\Big) \tag{5.1}$$

$$\simeq \Big(\mathbb{R}[p_i, p_i^\dagger, \hbar], \lambda_i p_i \frac{\delta}{\delta p_i^\dagger} + \frac{\hbar}{\lambda_i}\frac{\delta^2}{\delta p_i \delta p_i^\dagger}\Big) \tag{5.2}$$

Moreover, shifting the dual of the map $\mathcal{E}_\Gamma \simeq \ker\Delta_\Gamma \to \mathbb{R}^{\Gamma_0}$ extends to a map of Lie algebras:

$$\big(\mathbb{R}^{\Gamma_0}\big)^\vee[-1] \to \mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)$$

Upon which we can pplying Chevalley Eilenberg chains to obtain a map of coalgebras:

$$\mathrm{Sym}\big((\mathbb{R}^{\Gamma_0})^\vee\big) \to \mathrm{CE}_*\big(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)\big)$$

6. Homotopy Transfer/Renormalization Group Flow

We begin with a lemma, which gives an abstract way of comparing the above construction for homotopic pairings:

Lemma 9. *Given any* P_0 *with*

$$\mathrm{Q}_\Gamma \mathrm{P}_0 = \mathrm{K}_\Gamma - \mathrm{K}_0$$

There exists a map:

$$e^{\hbar \partial_{\mathrm{P}_0}} : \mathrm{CE}_*(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)) \to \mathrm{CE}_*(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_0))$$

6.1. **The Partition Function.** In particular, when $K_0 = 0$ and $K_0 = \mathrm{P}_\Gamma$ we obtain the map:

$$e^{\hbar \mathrm{P}_\Gamma} : \mathrm{CE}_*(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma))) \xrightarrow{\simeq} \mathrm{Sym}^\bullet(\mathcal{E}_\Gamma^\vee)[\hbar]$$

For example,

$$\mathrm{P}_\Gamma = \frac{1}{\lambda_i} \frac{\partial^2}{\partial p_i \partial p_i} = (\mathrm{Q}_\Gamma)^{-1}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

Where the sum is being taken over all nonzero eigenvalues, and Q_Γ first projects off the nullspace and then applies Q_Γ^{-1}

Remark 10. Note that we could have instead only projected off those eigenvectors with eigenvalue above a certain threshold, obtaining an "a low energy effective field theory".

Moreover, evalutating a closed cycle of $\mathcal{E}_\Gamma$ [2] yields a map of the form:

$$\mathrm{CE}_*(\mathrm{Heis}(\mathcal{E}_\Gamma, \omega_\Gamma)) \to \mathbb{R}[\hbar]$$

Precomposing with the inclusion of gives the partition function:

$$\mathcal{Z} : \mathrm{Sym}\big((\mathbb{R}^{\Gamma_0})^\vee\big) \to \mathbb{R}[\hbar]$$

One can check that the above map evaluates on some polynomial as:

$$\mathrm{e}^{(\mathrm{Q}_\Gamma)^{-1}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}} f(x)|_{x=0} = \int \mathrm{d}x \cdot e^{-\frac{1}{2} < x, Qx >} f(x)$$

which we can take the logarithm of to obtain a function:

$$\mathrm{S}^{\mathrm{eff}} = \log \mathcal{Z}$$

Which we'd either call the cumulant generating function, or the effective Hamiltonian/Action. The Taylor coefficients of this function may be computed Feynman diagrammatically.

Remark 11. Note this shouldn't be too surprising, as a Gaussian integral and (certain) retractions simply invert (certain) matrices.

[2] i.e. vector constant along each connected component

7. Appendix: Central Extensions

In this section, we will review the relationships between the Chevalley-Eilenberg construction and central extensions. We begin by fixing a Lie algbera $\mathfrak{g}$, and along with a map of k-modules:

$$\mathcal{U}_{\mathrm{Mod_k}}\mathrm{CE}_*(\mathfrak{g}) \xrightarrow{\eta} k[n]$$

In other words, an closed element of the Chevalley-Eilenberg cochains on $\mathfrak{g}$.

This map extends to a a map of coalgebras into the cofree commutative coalgebra generated by $k[n]$. Furthermore, this cofree commutative coalgebra is equivalent to the Chevalley-Eilenberg chains on the abelian Lie algebra generated by $k[n-1]$, so that the above is equivalent to:

$$\mathrm{CE}_*(\mathfrak{g}) \to \mathrm{CE}_*(k[n-1])$$

Differentiating this map gives a map of Lie algebras, which gives a fibre sequence of Lie algebras

$$\mathfrak{g}^\eta \to \mathfrak{g} \to k[n-1]$$

Note that the right adjointness of the abelian Lie algebra functor and the naturality of the pullback forces the fibre of $\mathfrak{g}^\eta \to \mathfrak{g}$ to be equivalent to the abelian Lie algebra $k[n-2]$. In other words, $\mathfrak{g}^\eta$ sits in the the middle of a fibre sequence of Lie algebras,

$$k[n-2] \to \mathfrak{g}^\eta \to \mathfrak{g}$$

Which is the definition of a central extension of Lie aglebras of (co)homological degree $(-)[n-2]$.

This further implies that there exists some an equivalence of k-modules:

$$\mathfrak{g}^\eta \simeq \mathfrak{g} \oplus k[n-2]$$

Fixing such a splitting of the above type decomposes and factors the bracket on $\mathfrak{g}^\eta$

$$\mathfrak{g}^\eta : \Lambda^\bullet(\mathfrak{g}^\eta[1]) \to \mathfrak{g}^\eta[2]$$

into:

$$\ell_{\mathfrak{g}^\eta} = \ell_{\mathfrak{g}} + \eta : \mathrm{Sym}^\bullet(\mathfrak{g}[1]) \to (\mathfrak{g} \oplus k[n-2])[2]) \simeq \mathfrak{g}[2] \oplus k[n]$$

In summary, we see that every closed element of (co)homological degree $(-)n$ Chevalley-Eilenberg cochains of a Lie algebra gives rise to a central extension of (co)homological degree $(-)[n-2]$.

7.1. **Example.** For example, let's take $\mathfrak{g}$ to be an ordinary Lie aglebra, and fix a map

$$\eta : \mathcal{U}_{\mathrm{Mod_k}}\mathrm{CE}_*(\mathfrak{g}) \to k[n]$$

In this case, $\mathrm{CE}_*(\mathfrak{g})$ is concentrated in strictly negative degree, so that the only interesting extensions will come from maps into $k[n]$ for $n \leq 1$. Moreover, for degree reasons, there must be a factorization of graded k-modules:

$$\mathcal{U}_{\mathrm{Mod_k}}(\mathrm{CE}_*(\mathfrak{g})) \to \mathrm{Sym}^n(\mathfrak{g}[1]) \to k[n]$$

If we want our central extension to a classical lie algebra, we need $n = 2$, so that the data is determined by a linear map:

$$\Lambda^2(\mathfrak{g}) \to k$$

For example, when $\mathfrak{g}$ is the lie algebra of the two dimensional torus, so that:

$$\mathrm{CE}^\bullet(\mathfrak{g}) \simeq \mathrm{H}^\bullet_{\mathrm{dR}}(\mathbb{T}^2) \simeq \mathbb{R}[dx, dp]$$

The above discussion therefore says that up to isomorphism, there exists a one dimensional vector space of central extensions.

A standard translation invariant symplectic form on the torus of gives a basis for this vector space. The central defines the classical Heisenberg Lie algebra:

$$[\partial_x, \partial_p] = \omega(\partial_x, \partial_p)\hbar = \{x, p\}\hbar$$

Where x, p are the coordinate functions around the identity arising as the Noether currents corresponding to the infinitesmal symmetries ∂_x, ∂_p.

A Lie algebraic model for the notion of quantization would say that a one-dimensional particle admits a unique quantization, up to a choice of $\hbar$. This constants may then be approximating fixed by performing an experiment.

As another example, we take take $\mathfrak{g}$ to be the lie algebra of $SU(2)$, in which case we see that the group of interest is:

$$\mathrm{H}^2\mathrm{CE}^*(\mathrm{su}(2)) \simeq \mathrm{H}^2_{\mathrm{dR}}(\mathrm{S}^3) \simeq 0$$

So that every central extension of degree zero su(2) is trivial. This means that every comoment map:

$$\mathfrak{g} \to \mathfrak{X}(M)$$

may be factored to give a global conserved current:

$$\mathfrak{g} \to \mathrm{C}^\infty(M)$$

in a manner compatible with the respecting Lie Structures.